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On the solvability of the Painlevé VI equation

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Abstract. A rigorous method was introduced by Fokas and Zhou for studying the Riemann–Hilbert problem associated with the Painlevé II and IV equations. The same methodology has been applied to the Painlevé I, III and V equations. In this paper, we will apply the same methodology to the Painlevé VI equation. We will show that the Cauchy problem for the Painlevé VI equation admits, in general, a global meromorphic solution in t. Furthermore, the special solution which can be written in terms of a hypergeometric function is obtained via solving the special case of the Riemann–Hilbert problem.

1. Introduction

At the beginning of this century Painlevé [13, 19] and his school [9] classified the equations of the form $y'' = F(y', y, z)$, where F is rational in y', algebraic in y and locally analytic in z, which have the Painlevé property; i.e. their solutions are free from movable critical points. Among fifty such equations, the six Painlevé equations are the most well known nonlinear ODEs, since they are irreducible and do not have the solutions in terms of the known functions. Besides the Painlevé property, these six Painlevé equations, PI–PVI, have mathematical and physical significance. Their mathematical importance originates from the following. (i) They can be considered as the isomonodromic conditions for suitable linear system of ODEs with rational coefficients possessing both regular and irregular singular points [8, 10, 2, 14]. (ii) They can be obtained as the similarity reduction of the nonlinear PDEs solvable by the inverse scattering transform (IST) [1]. For example, PI and PII can be obtained from the exact similarity reduction of the Korteweg–deVries (KdV) equation. (iii) For a certain choice of parameters, PII–PVI admit a one-parameter family of solutions which are either rational or can be expressed in terms of the classical transcendental functions. For example, PVI admits a one-parameter family of solutions in terms of hypergeometric functions [16, 3]. (iv) There are transformations associated with PII–PVI, these transformations map the solutions of a given Painlevé equation to the solution of the same equation but with different values of parameters [3, 17, 11, 12]. (v) PI–PV can be obtained from PVI by the process of contraction [13]. In a similar way, it is possible to obtain the associated transformations for PII–PIV from the transformation for PV. Moreover, the initial-value problem of the Painlevé equations (PI–PV) can be studied using the inverse monodromy transform (IMT) [4–7].

In this paper, we will apply the IMT method to PVI. This method is the extension of the inverse spectral transform from PDEs to ODEs, and can be thought of as a nonlinear analogue of Laplace's method to find the solution of linear ODEs. First important developments for

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studying the initial-value problem of Painlevé equations have been introduced by Flashka and Newell [2] and Jimbo *et al* [14]. They considered Painlevé equations as isomonodromic conditions for linear systems having both regular and irregular singular points. Solving such an initial-value problem is basically equivalent to solving an inverse problem for the associated isomonodromic linear equation. The inverse problem can be formulated in terms of the monodromy data which can be obtained from the initial data. In [2], this method is applied on PII and the special case of PIII, and the inverse problem is formulated in terms of a system of singular integral equations. In [14], the inverse problem is solved in terms of a formal infinite series uniquely determined in terms of the certain monodromy data. Ablowitz and Fokas [4] formulated the inverse problem for PII in terms of a matrix, singular, discontinuous Riemann–Hilbert (RH) boundary value problem defined on a complicated self-intersecting contour. Fokas and Zhou [6] introduce a rigorous methodology for studying the RH problem appearing in IMT, and they showed that the Cauchy problem for PII and PVI, in general, admit global solutions meromorphic in t. They also found the relation among the monodromy data (and hence, among the initial data) for which the solution is free from poles. In [7], the above rigorous methodology is applied to PI, PIII and PV.

The IMT method basically has the following two steps.

(i) *Direct problem.* The essence of the direct problem is to establish the analytic structure of the eigenfunction $Y(z,t)$ of one of the two associated linear problems in variable z. In the case of PVI, the linear ODE has regular singular points at $z = 0, 1, t, \infty$. Eigenfunctions normalized in the neighbourhood of the regular singular points $z = 0, 1, t$ are related with the eigenfunction in the neighbourhood of $z = \infty$ through the connection matrices. The set which consists of the entries of the connection matrices is called the set of the monodromy data. The crucial part of the direct problem is to show that only two of the monodromy data are arbitrary. This can be shown by using the product condition around all singular points (consistency condition) and certain equivalence relations. Hence, for given initial data for PVI the two independent monodromy data can be obtained.

(ii) *Inverse problem.* By using the results obtained from the direct problem a matrix RH problem can be formulated over a certain contour. The jump matrices for the RH problem are defined in terms of the monodromy data. The RH problem is discontinuous at the points of the discontinuities of the associated linear problem. These discontinuities can be avoided by inserting circles around the singularities. Now, the new RH problem is continuous and equivalent to the Fredholm integral equation. Once, the solution of the new RH problem is obtained the solution of the original one can easily be obtained.

Since, the eigenfunction $Y(z,t)$ is defined as the solution of the RH problem, once the solution of the RH problem is obtained the associated linear ODE can be used to obtain the solution of PVI. This procedure parametrizes the general solution of PVI in terms of the relevant monodromy data and shows that the general solution is meromorphic in t modulo the points $t = 0, 1, \infty$ which are its *critical points*. The *generalized* Cauchy data for $t = 0$ were introduced in [15]. In [15] an expression for the monodromy data in terms of the mentioned generalized Cauchy data was obtained. A combination of this result with the ones obtained in the present paper then provides the solution of the generalized Cauchy problem for the PVI equation.

As mentioned before, for a certain choice of the parameters PVI admits rational solutions as well as one-parameter families of solutions expressible in terms of a hypergeometric function. For special choices of the monodromy data the RH problem can be solved in a closed form. In the last section, as an example, we will show that for a particular choice of the monodromy data, the solution written in terms of the hypergeometric function can naturally be obtained by finding the closed-form solution of the RH problem. An exhaustive investigation of all such cases will be given elsewhere.

The sixth Painlevé equation

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right) \qquad (1.1)$$

can be obtained as the compatibility condition of the following linear system of equations [14]

$$\frac{\partial Y}{\partial z}=A(z,t)Y(z,t) \qquad (1.2a)$$
$$\frac{\partial Y}{\partial t}=B(z,t)Y(z,t) \qquad (1.2b)$$

where

$$A(z,t)=\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}=\begin{pmatrix} a_{11}(z,t) & a_{12}(z,t)\\ a_{21}(z,t) & a_{22}(z,t)\end{pmatrix}$$
$$A_i=\begin{pmatrix} u_i+\theta_i & -w_iu_i\\ w_i^{-1}(u_i+\theta_i) & -u_i\end{pmatrix}\qquad i=0,1,t \qquad B(z,t)=-A_t\frac{1}{z-t}. \qquad (1.3)$$

Setting

$$A_\infty=-(A_0+A_1+A_t)=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix}$$
$$\kappa_1+\kappa_2=-(\theta_0+\theta_1+\theta_t) \qquad \kappa_1-\kappa_2=\theta_\infty$$
$$a_{12}(z)=-\frac{w_0u_0}{z}-\frac{w_1u_1}{z-1}-\frac{w_tu_t}{z-t}=\frac{k(z-y)}{z(z-1)(z-t)} \qquad (1.4)$$
$$u=a_{11}(y)=\frac{u_0+\theta_0}{y}+\frac{u_1+\theta_1}{y-1}+\frac{u_t+\theta_t}{y-t}$$
$$\bar{u}=-a_{22}(y)=u-\frac{\theta_0}{y}-\frac{\theta_1}{y-1}-\frac{\theta_t}{y-t}.$$

Then

$$u_0+u_1+u_t=\kappa_2 \qquad w_0u_0+w_1u_1+w_tu_t=0$$
$$\frac{u_0+\theta_0}{w_0}+\frac{u_1+\theta_1}{w_1}+\frac{u_t+\theta_t}{w_t}=0 \qquad (1.5)$$
$$(t+1)w_0u_0+tw_1u_1+w_tu_t=k \qquad tw_0u_0=k(t)y$$

which are solved as

$$w_0 = \frac{ky}{tu_0} \qquad w_1 = -\frac{k(y-1)}{u_1(t-1)} \qquad w_t = \frac{k(y-t)}{t(t-1)u_t}$$

$$\begin{aligned}
u_0 &= \frac{y}{t\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [\theta_1(y-t) + t\theta_t(y-1) - 2\kappa_2(y-1)(y-t)]\bar{u} \\
&\qquad + \kappa_2^2(y-t-1) - \kappa_2(\theta_1 + t\theta_t)\} \\
u_1 &= -\frac{y-1}{(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [(\theta_1+\theta_\infty)(y-t) + t\theta_t(y-1) \\
&\qquad -2\kappa_2(y-1)(y-t)]\bar{u} + \kappa_2^2(y-t) - \kappa_2(\theta_1 + t\theta_t) - \kappa_1\kappa_2\} \\
u_t &= \frac{y-t}{t(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2 + [\theta_1(y-t) + t(\theta_t+\theta_\infty)(y-1) \\
&\qquad -2\kappa_2(y-1)(y-t)]\bar{u} + \kappa_2{}^2(y-1) - \kappa_2(\theta_1 + t\theta_t) - t\kappa_1\kappa_2\}.
\end{aligned} \tag{1.6}$$

The equation $Y_{zt} = Y_{tz}$ implies

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{y(y-1)(y-t)}{t(t-1)}\left(2u - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t - 1}{y-t}\right) \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{t(t-1)}\{[-3y^2 + 2(1+t)y - t]u^2 \\
&\qquad + [(2y-1-t)\theta_0 + (2y-t)\theta_1 + (2y-1)(\theta_t - 1)]u - \kappa_1(\kappa_2+1)\} \\
\frac{1}{k}\frac{\mathrm{d}k}{\mathrm{d}t} &= (\theta_\infty - 1)\frac{y-t}{t(t-1)}.
\end{aligned} \tag{1.7}$$

Thus y satisfies the sixth Painlevé equation (1.1), with the parameters

$$\alpha = \tfrac{1}{2}(\theta_\infty - 1)^2 \qquad \beta = -\tfrac{1}{2}\theta_0^2 \qquad \gamma = \tfrac{1}{2}\theta_1^2 \qquad \delta = \tfrac{1}{2}(1-\theta_t^2). \tag{1.8}$$

2. Direct problem

The essence of the direct problem is to establish the analytic structure of Y with respect to z, in the entire complex z-plane. Since (1.2a) is a linear ODE in z, therefore the analytic structure is completely determined by its singular points. The equation (1.2a) has regular singular points at $z = 0, 1, t, \infty$.

2.1. Solution about $z = 0$

It is well known that if the coefficient matrix of the linear ODE has an isolated singularity at $z = 0$, then the solution in the neighbourhood of $z = 0$ can be obtained via a convergent power series. In this particular case the solution $Y_0(z) = (Y_{0(1)}(z), Y_{0(2)}(z))$, for $\theta_0 \neq n,\ n \in \mathbb{Z}$ has the form

$$Y_0(z) = \hat{Y}_0(z)z^{D_0} = G_0(I + \hat{Y}_{01}z + \hat{Y}_{02}z^2 + \cdots)z^{D_0} \qquad |z| < 1 \tag{2.1}$$

where $\hat{Y}_0(z)$ is holomorphic at $z = 0$ and,

$$G_0 = \begin{pmatrix} 2k_0 & l_0 w_0 u_0 \\ 2\frac{k_0}{w_0} & l_0(u_0 + \theta_0) \end{pmatrix} \qquad \det G_0 = 1 \qquad D_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

$$k_0 = \tilde{k}_0 \, e^{\sigma_0(t)} \qquad l_0 = \tilde{l}_0 \, e^{-\sigma_0(t)} \qquad \tilde{k}_0, \tilde{l}_0 = \text{constant}$$
$$\sigma_0 = \int^t \frac{1}{t'} \left[u_t + \theta_t - \frac{w_t u_t}{w_0} \right] dt' \tag{2.3}$$

and $\hat{Y}_{01}$ satisfies the following equation:

$$\hat{Y}_{01} + [\hat{Y}_{01}, D_0] = -G_0^{-1} \left(A_1 G_0 - \frac{dG_0}{dt} \right). \tag{2.4}$$

For simplicity in the notation the t dependence is suppressed. Equation (2.3) follows from that $Y_0(z)$ also satisfies (1.2b) and $\det \hat{Y}_0(z) = 1$. If $\theta_0 = n$, $n \in \mathbb{Z}$ then the solution $Y_0(z)$ may or may not have the $\log z$ term.

The monodromy matrix about $z = 0$ is given as

$$Y_0(z \, e^{2i\pi}) = Y_0(z) \, e^{2i\pi D_0}. \tag{2.5}$$

2.2. *Solution about z = 1*

The solution $Y_1(z) = (Y_{1(1)}(z), Y_{1(2)}(z))$, of (1.2) in the neighbourhood of the regular singular point $z = 1$ for $\theta_1 \neq n$, $n \in \mathbb{Z}$ has the form

$$Y_1(z) = \hat{Y}_1(z)(z-1)^{D_1} = G_1(I + \hat{Y}_{11}(z-1) + \hat{Y}_{12}(z-1)^2 + \cdots)(z-1)^{D_1}$$
$$|z - 1| < 1 \tag{2.6}$$

where $\hat{Y}_1(z)$ is holomorphic at $z = 1$ and

$$G_1 = \begin{pmatrix} 2k_1 & l_1 w_1 u_1 \\ 2\dfrac{k_1}{w_1} & l_1(u_1 + \theta_1) \end{pmatrix} \qquad \det G_1 = 1 \qquad D_1 = \begin{pmatrix} \theta_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.7}$$

$$k_1 = \tilde{k}_1 \, e^{\sigma_1(t)} \qquad l_1 = \tilde{l}_1 \, e^{-\sigma_1(t)} \qquad \tilde{k}_1, \tilde{l}_1 = \text{constant}$$
$$\sigma_1 = \int^t \frac{1}{t'-1} \left[u_t + \theta_t - \frac{w_t u_t}{w_1} \right] dt' \tag{2.8}$$

and $\hat{Y}_{11}$ satisfies the following equation:

$$\hat{Y}_{11} + [\hat{Y}_{11}, D_1] = G_1^{-1} \left(A_0 G_1 - \frac{dG_1}{dt} \right). \tag{2.9}$$

Equation (2.8) follows from the fact that $Y_1(z)$ also solves (1.2b) and $\det \hat{Y}_1(z) = 1$. If $\theta_1 = n$, $n \in \mathbb{Z}$, the solution $Y_1(z)$ may or may not contain the $\log(z-1)$ term.

The monodromy matrix about $z = 1$ is given as

$$Y_1(z \, e^{2i\pi}) = Y_1(z) \, e^{2i\pi D_1}. \tag{2.10}$$

2.3. *Solution about z = t*

The solution $Y_t(z) = (Y_{t(1)}(z), Y_{t(2)}(z))$, of (1.2) in the neighbourhood of the regular singular point $z = t$ for $\theta_t \neq n$, $n \in \mathbb{Z}$ (if $\theta_t = n$, $n \in \mathbb{Z}$ the solution $Y_t(z)$ may or may not have the $\log(z - t)$ term) has the form

$$Y_t(z) = \hat{Y}_t(z)(z-t)^{D_t} = G_t(I + Y_{t1}(z-t) + Y_{t2}(z-t)^2 + \cdots)(z-t)^{D_t} \qquad |z - t| < 1 \tag{2.11}$$

where $\hat{Y}_t(z)$ is holomorphic at $z = t$ and

$$G_t = \begin{pmatrix} 2k_t & l_t w_t u_t \\ 2\frac{k_t}{w_t} & l_t(u_t + \theta_t) \end{pmatrix} \qquad \det G_t = 1 \qquad D_t = \begin{pmatrix} \theta_t & 0 \\ 0 & 0 \end{pmatrix} \tag{2.12}$$

$$k_t = \tilde{k}_t \, e^{\sigma_t(t)} \qquad l_t = \tilde{l}_t \, e^{-\sigma_t(t)} \qquad \tilde{k}_t, \tilde{l}_t = \text{constant}$$

$$\sigma_t = \int^t \left[\frac{1}{t'} \left(u_0 + \theta_0 - \frac{w_0 u_0}{w_t} \right) + \frac{1}{t'-1} \left(u_1 + \theta_1 - \frac{w_1 u_1}{w_t} \right) \right] dt' \tag{2.13}$$

and $\hat{Y}_{t1}$ satisfies the following equation:

$$\hat{Y}_{t1} + [\hat{Y}_{t1}, D_t] = G_t^{-1} \frac{d G_t}{dt}. \tag{2.14}$$

Equation (2.13) follows from the fact that the solution $Y_t(z)$ also satisfy (1.2*b*) and $\det Y_t(z) = 1$.

The monodromy matrix about $z = t$ is given as

$$Y_t(z \, e^{2i\pi}) = Y_t(z) \, e^{2i\pi D_t}. \tag{2.15}$$

2.4. *Solution about* $z = \infty$

The solution $Y(z) = (Y_{(1)}(z), Y_{(2)}(z))$, of (1.2) in the neighbourhood of the regular singular point $z = \infty$ for $\theta_\infty \neq n$, $n \in \mathbb{Z}$ (if $\theta_\infty = n$, $n \in \mathbb{Z}$, the solution may or may not have the $\log(\frac{1}{z})$ term) has the form

$$Y(z) = \hat{Y}_\infty(z) \left(\frac{1}{z} \right)^{D_\infty} = \left(I + \hat{Y}_{\infty 1} \frac{1}{z} + \hat{Y}_{\infty 2} \left(\frac{1}{z} \right)^2 + \cdots \right) \left(\frac{1}{z} \right)^{D_\infty} \qquad z \to \infty \tag{2.16}$$

where $\hat{Y}(z)$ is holomorphic at $z = \infty$ and

$$D_\infty = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \tag{2.17}$$

$$\kappa_2 = u_0 + u_1 + u_t \qquad \kappa_1 - \kappa_2 = \theta_\infty \qquad \kappa_1 + \kappa_2 = -(\theta_0 + \theta_1 + \theta_t)$$

and $\hat{Y}_{\infty 1}$ satisfies the following equation:

$$\hat{Y}_{\infty 1} + [\hat{Y}_{\infty 1}, D_\infty] = -(A_1 + t A_t). \tag{2.18}$$

The monodromy matrix about $z = \infty$ is given as

$$Y(z \, e^{2i\pi}) = Y(z) \, e^{-2i\pi D_\infty}. \tag{2.19}$$

We associate the branch cuts from 0 to 1 and from 1 to t with z^{D_0} and $(z-1)^{D_1}$ respectively, while the branch cut from t to ∞ with $(z-t)^{D_t}$ and $(1/z)^{D_\infty}$ as indicated in figure 1.

2.5. *Monodromy data*

The relations between the $Y(z)$ and $Y_i(z)$, $i = 0, 1, t$ are given by the connection matrices E_i,

$$Y(z) = Y_i(z) E_i \qquad E_i = \begin{pmatrix} \mu_i & \nu_i \\ \zeta_i & \eta_i \end{pmatrix} \qquad \det E_i = 1 \qquad i = 0, 1, t. \tag{2.20}$$

Since, $Y(z)$ and $Y_i(z)$, $i = 0, 1, t$ satisfy (1.2*a*), they are related with constant matrices E_i with respect to z and $\det E_1 = 1$ condition follows from the normalization of $Y_i(z)$ to have unit determinant.

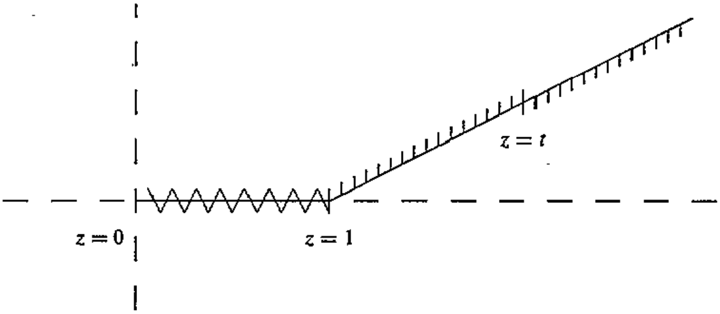

Figure 1.

The monodromy data $MD = \{\mu_0, \nu_0, \zeta_0, \eta_0, \mu_1, \nu_1, \zeta_1, \eta_1, \mu_t, \nu_t, \zeta_t, \eta_t\}$ satisfy the following consistency condition:

$$(E_t^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_t}E_t)(E_1^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_1}E_1)(E_0^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_0}E_0) = \mathrm{e}^{-2\mathrm{i}\pi D_\infty} \tag{2.21}$$

in particular,

$$\begin{aligned}\cos\pi(\theta_0-\theta_1)(\zeta_0\mu_0\eta_1\nu_1+\eta_0\nu_0\mu_1\zeta_1-\eta_0\mu_0\nu_1\zeta_1-\zeta_0\nu_0\eta_1\mu_1)\\ +\cos\pi(\theta_0+\theta_1)(\nu_0\zeta_0\nu_1\zeta_1+\eta_0\mu_0\eta_1\mu_1-\mu_0\zeta_0\nu_1\eta_1-\eta_0\nu_0\mu_1\zeta_1)\\ =\mu_t\eta_t\cos\pi(\theta_\infty+\theta_t)-\nu_t\zeta_t\cos\pi(\theta_\infty-\theta_t)\,.\end{aligned} \tag{2.22}$$

It is possible to show that only two of the monodromy data (two entries of the connection matrix E_0) are arbitrary and all the others can be determined in terms of these two. If we let [7]

$$E_1(E_0^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_0}E_0)\,\mathrm{e}^{2\mathrm{i}\pi D_\infty}E_1^{-1} = \begin{pmatrix} x & \tau \\ -\frac{1}{\tau}(x^2-cx+1) & c-x\end{pmatrix}\mathrm{e}^{-\mathrm{i}\pi(\theta_1+\theta_t)} \tag{2.23}$$

then the consistency condition (2.21) gives

$$E_1(E_t^{-1}\,\mathrm{e}^{-2\mathrm{i}\pi D_t}E_t)E_1^{-1} = \mathrm{e}^{2\mathrm{i}\pi D_1}\begin{pmatrix} x & \tau \\ -\frac{1}{\tau}(x^2-cx+1) & c-x\end{pmatrix}\mathrm{e}^{-\mathrm{i}\pi(\theta_1+\theta_t)}\,. \tag{2.24}$$

The trace of (2.23) and (2.24) imply

$$\mu_0\eta_0\cos\pi(\theta_0+\theta_\infty)-\nu_0\zeta_0\cos\pi(\theta_0-\theta_\infty)=\frac{c}{2}\qquad 2\cos\pi\theta_t = c\,\mathrm{e}^{-\mathrm{i}\pi\theta_1}+2\mathrm{i}\,x\sin\pi\theta_1\,. \tag{2.25}$$

Thus, x and c can be determined in terms of the entries of the connection matrix E_0, if $\theta_1 \neq n$, $n \in \mathbb{Z}$. τ is the only free parameter in (2.23), which reflects the freedom in choosing the connection matrix E_1, i.e. E_1 can be determined within the left multiplicative diagonal matrix $\mathrm{diag}\,(d_1, d_1^{-1})$, where d_1 is non-zero arbitrary complex constant. If we replace E_1 by $\mathrm{diag}\,(d_1, d_1^{-1})E_1$ in (2.23), this changes τ to τ/d_1^2. But, this transformation in E_1 leaves the consistency condition (2.21) invariant. Also the consistency condition (2.21) remains the same if E_t is replaced by $\mathrm{diag}(d_t, d_t^{-1})E_t$, where d_t is an arbitrary non-zero complex constant. Hence, equation (2.24) determines E_t within the left multiplicative diagonal matrix $\mathrm{diag}(d_t, d_t^{-1})$. On the other hand, if we replace Y with $\tilde{Y} = R^{-1}YR$ in (1.2) where $R = \mathrm{diag}(r^{1/2}, r^{-1/2})$ and r is non-zero arbitrary complex constant, equation (1.2) for $\tilde{Y}$ is the same as for Y, with the only change replacing w_i with w_i/r, $i = 0, 1, t$. The solution $y(t)$ of PVI does not change under this transformation (see the last equation of (1.5)). But, the connection matrix $\tilde{E}_0$ for $\tilde{Y}$ is obtained by replacing ν_0 and ζ_0 with

ν_0/r and $\zeta_0 r$, respectively. Thus, r may be chosen to eliminate one of the entries of E_0, e.g. $r = \nu_0$. Also, changing the arbitrary integration constants in $\sigma_0(t)$ (see equation (2.3)) amounts to multiplying $Y_{0(1)}(z)$ and $Y_{0(2)}(z)$ by arbitrary non-zero complex constants d_0 and d_0^{-1} respectively. This maps E_0 to $\mathrm{diag}\,(d_0, d_0^{-1})E_0$. Thus, d_0 may be chosen to eliminate one of the entries of the connection matrix E_0.

The freedom in choosing $E_i,\ i = 0, 1, t$ does not effect the solution of the RH-problem. Equation (2.20a) and the above transformations ($E_i \to \mathrm{diag}\,(d_i, d_i^{-1})E_i,\ i = 0, 1, t$) change Y_i to $Y_i\mathrm{diag}\,(d_i, d_i^{-1})$, i.e. the transformations have the effect of transforming k_i to $k_i d_i$ and l_i to l_i/d_i, $i = 0, 1, t$, which leaves $k_i l_i = 1/2\theta_i$ ($\det G_i = 1$) invariant.

By using the similar proofs given in [2, 7] it is possible to prove that, if Y evolves in t according to (1.2b), then the monodromy data are independent of t.

3. The inverse problem

In this section, we will formulate a continuous, regular RH problem over the self-intersecting contour for the function called $\Phi(z)$. In order to have a regular RH problem, we let $0 \leqslant \theta_i < 1$, $i = 0, 1, t, \infty$. The general case can be obtained by using the Schlesinger transformations for PVI [18]. Since, $\hat{Y}_i(z)$, $i = 0, 1, t$ and $\hat{Y}(z)$ are holomorphic at $z = 0, 1, t, \infty$, respectively, we first consider the contour indicated in figure 2 instead of figure 1 to formulate the continuous RH problem. The circles about $z = 0, 1, t$ have radius $r < \frac{1}{2}$ and are denoted by C_0, C_1 and C_t, respectively.

The jumps across C_0, $\widehat{CD}$, $\widehat{EF}$ are given by the connection matrices E_0, E_1 and E_t, respectively. All the other jumps across the rest of the contour can be derived from the definition of the connection matrices and the monodromy matrices. To drive jump across $\overline{BC}$, we use the definition of the connection matrix E_0 and (2.5):

$$\begin{aligned} Y(z) &= Y_0(z)E_0 \\ &= Y_0(z\,\mathrm{e}^{2\mathrm{i}\pi})\,\mathrm{e}^{-2\mathrm{i}\pi D_0}E_0 \\ &= Y(z\,\mathrm{e}^{2\mathrm{i}\pi})E_0^{-1}\,\mathrm{e}^{-2\mathrm{i}\pi D_0}E_0\,. \end{aligned} \tag{3.1}$$

The jump across $\underset{\smile}{CD}$, can be obtained from (3.1) and the definition of the connection matrix E_1:

$$Y(z) = Y_1(z\,\mathrm{e}^{2\mathrm{i}\pi})E_1(E_0^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_0}E_0) \tag{3.2}$$

since, $Y_1(z)$ is holomorphic at $z = 0$, jump across the $\underset{\smile}{CD}$ is given as

$$Y(z) = Y_1(z)E_1(E_0^{-1}\,\mathrm{e}^{2\mathrm{i}\pi D_0}E_0)\,. \tag{3.3}$$

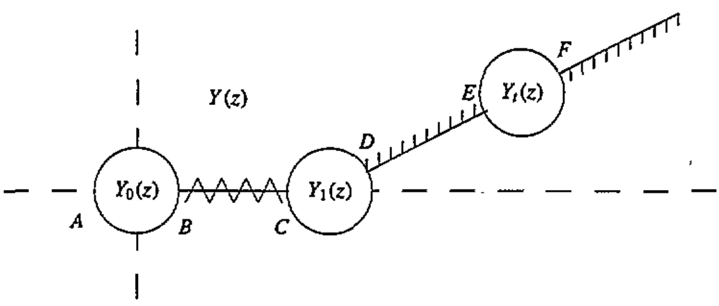

Figure 2.

The jump across $\overline{DE}$:

$$\begin{aligned} Y(|z-1|) &= Y_1(|z-1|)E_1 \\ &= Y_1(|z-1|\,e^{2i\pi})\,e^{-2i\pi D_1}E_1 \\ &= Y(|z-1|\,e^{2i\pi})(E_0^{-1}\,e^{-2i\pi D_0}E_0)(E_1^{-1}\,e^{-2i\pi D_1}E_1)\,. \end{aligned} \tag{3.4}$$

In a similar way the jumps across the contours $\underset{\smile}{EF}$ and $\overline{F\infty}$ can be derived. Hence, the jumps across the contours of figure 2 are given by

$$\begin{aligned} &C_0: \; Y(z) = Y(z)E_0 \\ &\overline{BC}: Y(z) = Y(z\,e^{2i\pi})E_0^{-1}\,e^{-2i\pi D_0}E_0 \\ &\widehat{CD}: \; Y(z) = Y_1(z)E_1 \\ &\underset{\smile}{CD}: Y(z) = Y_1(z)E_1E_0^{-1}\,e^{2i\pi D_0}E_0 \\ &\overline{DE}: Y(|z-1|) = Y(|z-1|e^{2i\pi})(E_0^{-1}\,e^{-2i\pi D_0}E_0)(E_1^{-1}\,e^{-2i\pi D_1}E_1) \\ &\widehat{EF}: \; Y(z) = Y_t(z)E_t \\ &\underset{\smile}{EF}: Y(|z-t|) = Y_t(|z-t|)\,e^{-2i\pi D_t}E_t\,e^{-2i\pi D_\infty} \\ &\overline{F\infty}: Y(z) = Y(z\,e^{2i\pi})\,e^{2i\pi D_\infty}\,. \end{aligned} \tag{3.5}$$

In order to define the continuous RH problem, we define sectionally analytic function $\Phi(z,t)$ as follows:

$$\begin{aligned} Y(z) &= \Phi(z)\left(\frac{1}{z}\right)^{D_\infty} \qquad & Y_0(z) &= \Phi_0(z)z^{D_0} \\ Y_1(z) &= \Phi_1(z)(z-1)^{D_1} \qquad & Y_t(z) &= \Phi_t(z)(z-t)^{D_t}\,. \end{aligned} \tag{3.6}$$

The orientation used in figure 3 allows the splitting of the complex z-plane in $+$ and $-$ regions. Then, $\Phi^{\pm}$, Φ_i, $i = 0, 1, t$ are the representations of the sectionally analytic function $\Phi(z)$ in the regions indicated in figure 3. Equation (3.5) implies certain jumps for $\Phi(z)$ and we obtain the following RH problem:

$$\Phi^+(\hat{z}) = \Phi^-(\hat{z})V(\hat{z}) \qquad \text{on} \quad C \qquad \Phi(z) = I + O\left(\frac{1}{z}\right) \qquad \text{as} \quad z \to \infty \tag{3.7}$$

where $C = \overline{\infty A} + C_0 + \overline{BC} + C_1 + \overline{DE} + C_t + \overline{E\infty}$ and the jump matrices are given by

$$V_{\overline{\infty A}} = I \qquad V_{\widehat{AB}} = z^{D_0}E_0\left(\frac{1}{z}\right)^{-D_\infty}$$

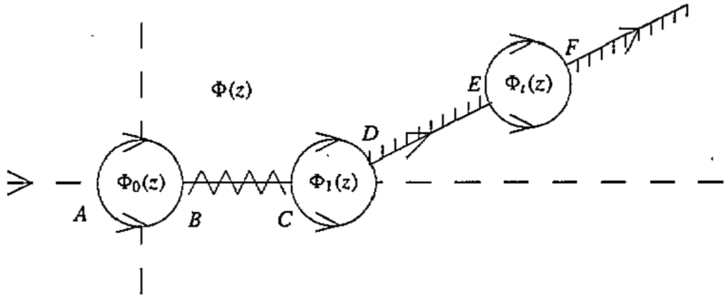

Figure 3.

$$V_{\underset{\smile}{AB}} = \left(\frac{1}{z}\right)^{D_\infty} E_0^{-1} z^{-D_0} \qquad V_{\overline{BC}} = \left(\frac{1}{z}\right)^{D_\infty} E_0^{-1}\, \mathrm{e}^{-2i\pi D_0} E_0 \left(\frac{1}{z}\right)^{-D_\infty}$$

$$V_{\widehat{CD}} = (z-1)^{D_1} E_1 \left(\frac{1}{z}\right)^{-D_\infty} \qquad V_{\underset{\smile}{CD}} = \left(\frac{1}{z}\right)^{D_\infty} E_0^{-1}\, \mathrm{e}^{-2i\pi D_0} E_0 E_1^{-1} (z-1)^{-D_1} \tag{3.8}$$

$$V_{\overline{DE}} = \left(\frac{1}{z}\right)^{D_\infty} (E_0^{-1}\, \mathrm{e}^{-2i\pi D_0} E_0)(E_1^{-1}\, \mathrm{e}^{-2i\pi D_1} E_1) \left(\frac{1}{z}\right)^{-D_\infty} \qquad V_{\widehat{EF}} = (z-t)^{D_t} E_t \left(\frac{1}{z}\right)^{-D_\infty}$$

$$V_{\underset{\smile}{EF}} = \left(\frac{1}{z}\right)^{D_\infty} \mathrm{e}^{2i\pi D_\infty} E_t^{-1}\, \mathrm{e}^{-2i\pi D_t} (z-t)^{-D_t} \qquad V_{\overline{F\infty}} = \left(\frac{1}{z}\right)_+^{D_\infty} \mathrm{e}^{2i\pi D_\infty} \left(\frac{1}{z}\right)^{-D_\infty}.$$

The subscript + in $V_{\overline{F\infty}}$ denotes that we consider the boundary value from the + region, i.e. $(z)_+ = |z|\, \mathrm{e}^{2i\pi}$.

By construction $\Phi(z)$ satisfies the continuous RH problem and this can be checked by the product of the jump matrices V at the intersection points. The product conditions give

$$\begin{aligned} &A:\ V_{\widehat{AB}} V_{\underset{\smile}{AB}} = I && B:\ [V_{\underset{\smile}{AB}}]_+ V_{\widehat{AB}} [V_{\overline{BC}}]^{-1} = I \\ &C:\ [V_{\overline{BC}}]^{-1} V_{\underset{\smile}{CD}} V_{\widehat{CD}} = I && D:\ [V_{\underset{\smile}{CD}}]_+ V_{\widehat{CD}} [V_{\overline{DE}}]^{-1} = I \\ &E:\ [V_{\overline{DE}}]^{-1} V_{\underset{\smile}{EF}} V_{\widehat{EF}} = I && F:\ [V_{\overline{F\infty}}]^{-1} [V_{\underset{\smile}{EF}}]_+ V_{\widehat{EF}} = I\,. \end{aligned} \tag{3.9}$$

The product conditions at the intersection points A, B, C, D and F are satisfied identically and the product condition at point E is satisfied because of the consistency condition (2.21) of the monodromy data. In equation (3.9b), $[V_{\underset{\smile}{AB}}]_+$ indicates that z term in $V_{\underset{\smile}{AB}}$ must be evaluated as $(z)_+$, in equation (3.9d), $[V_{\underset{\smile}{CD}}]_+$ indicates that $(z-1)$ term must be evaluated as $(z-1)_+$ and in equation (3.9f), $\left(\frac{1}{z}\right)$ and $(z-t)$ terms in $V_{\underset{\smile}{EF}}$ must be evaluated as $\left(\frac{1}{z}\right)_+$ and $(z-t)_+$, respectively.

The RH problem (3.7) is equivalent to following Fredholm integral equation

$$\Phi^-(z) = I + \frac{1}{2i\pi} \int_C \frac{\Phi^-(\hat{z})[V(\hat{z})V^{-1}(z) - I]}{\hat{z} - z}\, \mathrm{d}\hat{z}\,. \tag{3.10}$$

3.1. Derivation of the linear problem

In this section, we will show that if the sectionally analytic function $\Phi(z)$ satisfying the RH problem (3.7) is known, then the coefficients A and B of the Lax pair can be determined and hence the solution of PVI.

We define A by $A(z) = \frac{\partial Y}{\partial z} Y^{-1}(z)$. Since $\frac{\partial Y}{\partial z}$ and $Y(z)$ admit the same jumps it follows that $A(z)$ is holomorphic in $\mathbb{C}/(\{0\}+\{1\}+\{t\})$ and has the removable singularity at $z = \infty$. Furthermore, $Y(z) \sim (1/z)^{D_\infty}$ as $z \to \infty$, and thus $A(z) = A_0\frac{1}{z} + A_1\frac{1}{z-1} + A_t\frac{1}{z-t}$. Recall that, $Y(z)$ and $\Phi(z)$ are related via (3.6), therefore (3.6) and $\frac{\partial Y}{\partial z} = A(z)Y(z)$ give

$$\begin{aligned} &\frac{\partial \Phi}{\partial z} - \Phi D_\infty \frac{1}{z} = \left[A_0 \frac{1}{z} + A_1 \frac{1}{z-1} + A_t \frac{1}{z-t}\right]\Phi && \text{near} \quad z = \infty \\ &\frac{\partial \Phi}{\partial z} + \Phi D_0 \frac{1}{z} = \left[A_0 \frac{1}{z} + A_1 \frac{1}{z-1} + A_t \frac{1}{z-t}\right]\Phi && \text{near} \quad z = 0 \\ &\frac{\partial \Phi}{\partial z} + \Phi D_1 \frac{1}{z-1} = \left[A_0 \frac{1}{z} + A_1 \frac{1}{z-1} + A_t \frac{1}{z-t}\right]\Phi && \text{near} \quad z = 1 \\ &\frac{\partial \Phi}{\partial z} + \Phi D_t \frac{1}{z-t} = \left[A_0 \frac{1}{z} + A_1 \frac{1}{z-1} + A_t \frac{1}{z-t}\right]\Phi && \text{near} \quad z = t\,. \end{aligned} \tag{3.11}$$

For large z, $\Phi(z)$ has the expansion

$$\Phi(z) = I + \Phi_{-1}\frac{1}{z} + \Phi_{-2}\frac{1}{z^2} + O\left(\frac{1}{z^3}\right) \qquad \text{as} \quad z \to \infty . \tag{3.12}$$

Substituting (3.12) into (3.11a) yields

$$O\left(\frac{1}{z}\right): \; A_0 + A_1 + A_t = -D_\infty \qquad O\left(\frac{1}{z^2}\right): \; -\Phi_{-1} + [D_\infty, \Phi_{-1}] = A_1 + tA_t . \tag{3.13}$$

Since, the function $\Phi(z)$ is sectionally analytic and $\Phi(z) = \Phi_i(z)$, $i = 0, 1, t$ about $z = 0, 1, t$, respectively, then (3.11b)–(d) imply

$$A_0 = \Phi_0(0)D_0\Phi_0^{-1}(0) \qquad A_1 = \Phi_1(1)D_1\Phi_1^{-1}(1) \qquad A_t = \Phi_t(t)D_t\Phi_t^{-1}(t) \tag{3.14}$$

respectively. Thus,

$$\det A_i = 0 \qquad \operatorname{trace} A_i = \theta_i \qquad i = 0, 1, t . \tag{3.15}$$

The equations (3.13) and (3.15) imply that A_i, $i = 0, 1, t$ can be taken in the form appearing in (1.3b), then (3.13b) gives

$$(\Phi_{-1})_{12}(1 - \theta_\infty) = u_1 w_1 + t u_t w_t = -k(t) . \tag{3.16}$$

Hence, the solution $y(t)$ of PVI can be written in terms of $(\Phi_{-1})_{12}$.

Similarly consideration implies that B is holomorphic in $\mathbb{C}/\{t\}$ and has removable singularity at $z = \infty$. Thus $B(z) = B_0\frac{1}{z-t}$. Using $\frac{\partial Y}{\partial t} = BY$ and (3.6) it follows that

$$\begin{aligned} &\frac{\partial \Phi}{\partial t} = B_0\frac{1}{z-t}\Phi \qquad \text{near} \quad z = \infty \\ &\frac{\partial \Phi}{\partial t} - \frac{1}{z-t}\Phi D_t = B_0\frac{1}{z-t}\Phi \qquad \text{near} \quad z = t . \end{aligned} \tag{3.17}$$

These equations imply

$$\frac{\partial \Phi_{-1}}{\partial t} = B_0 \qquad B_0 = -\Phi_t(t)D_t\Phi_t^{-1}(t) \tag{3.18}$$

respectively. Equations (3.18b) imply that B_0 can be taken as $B_0 = -A_t$. Equation (3.18a) with $(\Phi_{-1})_{12}$ is consistent with the compatibility condition of (1.2b).

4. Closed-form solution

For certain choice of the parameters, PVI admit one parameter family of solutions which are expressible in terms of hypergeometric function [16, 3]. In this section, we will show that, for certain choice of the monodromy data such solution can naturally be obtained by finding the closed-form solution of the RH problem (3.7).

Let $\nu_0 = \zeta_0 = \zeta_1 = 0$, then the consistency condition (2.21) implies that $\zeta_t = 0$, $\theta_0 + \theta_1 + \theta_t + \kappa_1 = p$, and $\kappa_2 = q$, $p, q \in \mathbb{Z}$. Without loss of generality, we let $E_0 = I$, and $\mu_1 = \eta_1 = \mu_t = \eta_t = 1$, $p = q = 0$. Then the matrix valued RH problem can be reduced to set of scaler RH problems. If $\Phi(z) = (\Phi_{(1)}(z), \Phi_{(2)}(z))$, then

$$\Phi^+_{(1)}(\hat{z}) = \Phi^-_{(1)}(\hat{z})g(\hat{z}) \qquad \Phi^+_{(2)}(\hat{z}) - \Phi^-_{(2)}(\hat{z}) = h(\hat{z})\Phi^-_{(1)}(\hat{z}) \qquad \text{on} \quad C \tag{4.1}$$

where the jump functions $g(\hat{z})$ and $h(\hat{z})$ can be obtained from (3.8) for this particular choice. The RH problem for $\Phi_{(1)}(z)$ can easily be solved by introducing new sectionally analytic function $\Psi(z)$ such that $\Phi_{(1)}(z) = \Psi(z)z^{\theta_0+\theta_\infty}(z-1)^{\theta_1}(z-t)^{\theta_t}$, $\Phi_{0(1)}(z) =$

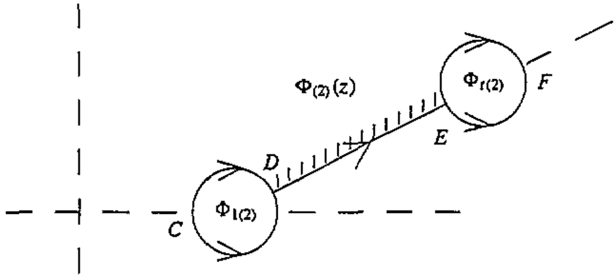

Figure 4.

$\Psi_0(z)(z-1)^{\theta_1}(z-t)^{\theta_t}$, $\Phi_{1(1)}(z) = \Psi_1(z)z^{\theta_0}(z-t)^{\theta_t}$, $\Phi_{t(1)}(z) = \Psi_t(z)z^{\theta_0}(z-1)^{\theta_1}$. Then, $\Psi(z)$ satisfies the following RH problem:

$$\Psi^+(\hat{z}) = \Psi^-(\hat{z}) \qquad \text{on} \quad C \qquad \Psi(z) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{as} \quad z \to \infty \tag{4.2}$$

i.e. $\Psi^+(z)$ and $\Psi^-(z)$ are analytic continuation of each other. Thus,

$$\begin{aligned} \Phi_{(1)}(z) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{\theta_0+\theta_\infty}(z-1)^{\theta_1}(z-t)^{\theta_t} \qquad & \Phi_{0(1)}(z) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}(z-1)^{\theta_1}(z-t)^{\theta_t} \\ \Phi_{1(1)}(z) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{\theta_0}(z-t)^{\theta_t} & \Phi_{t(1)}(z) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{\theta_0}(z-1)^{\theta_1}. \end{aligned} \tag{4.3}$$

Hence, the RH problem for $\Phi_{(2)}(z)$:

$$\begin{aligned} &\Phi^+_{(2)}(\hat{z}) - \Phi^-_{(2)}(\hat{z}) = k(\hat{z}) \qquad \text{on} \quad C_{(2)} = C_1 + \overline{DE} + C_t \\ &\Phi_{(2)}(z) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{as} \quad z \to \infty \end{aligned} \tag{4.4}$$

where $C_{(2)}$ is indicated in figure 4, and the jump functions $k(z)$ are

$$\begin{aligned} &k_{\widehat{CD}} = -k_{\underset{\smile}{CD}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_1 r(z) \qquad k_{\overline{DE}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_1 r(z)(1 - \mathrm{e}^{2\mathrm{i}\pi\theta_1}) \\ &k_{\widehat{EF}} = -k_{\underset{\smile}{EF}}\,\mathrm{e}^{2\mathrm{i}\pi\theta_t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_t r(z) \qquad r(z) = z^{\theta_0}(z-1)^{\theta_1}(z-t)^{\theta_t}. \end{aligned} \tag{4.5}$$

By Plemelj's formula the solution of the RH problem for $\Phi_{(2)}(z)$ is given as

$$\Phi^+_{(2)}(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\mathrm{i}\pi}\int_{C_{(2)}} \frac{k(\hat{z})}{\hat{z} - z}\,\mathrm{d}\hat{z}. \tag{4.6}$$

Evaluating the integrals over the contours C_1 and C_t and using the consistency condition of the monodromy data, $\Phi_{(2)}(z)$ is obtained as follows:

$$\begin{aligned} \Phi^+_{(2)}(z) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\mathrm{i}\pi}\begin{pmatrix} 1 \\ 0 \end{pmatrix} F(z,t) \\ F(z,t) &= \frac{\nu_1}{2\mathrm{i}\pi}(1 - \mathrm{e}^{2\mathrm{i}\pi\theta_1})\int_1^t \frac{r(\hat{z})}{\hat{z} - z}\,\mathrm{d}\hat{z} \end{aligned} \tag{4.7}$$

where $r(z)$ is given in (4.5). Combining (4.3a) with (4.7) and using (3.6a) yield

$$Y(z) = \begin{pmatrix} z^{\theta_0}(z-1)^{\theta_1}(z-t)^{\theta_t} & F(z,t) \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

Expanding $F(z,t)$ for large z the coefficient $f(t)$ of the $O\left(\frac{1}{z}\right)$ term gives (see equation (1.5*d*))

$$k(t) = (\theta_\infty - 1)\,f(t) \tag{4.9}$$

and expanding $F(z,t)$ in powers of z the coefficient $f_0(t)$ of $O(1)$ term gives

$$u_0 w_0 = \theta_0 f_0(t)\,. \tag{4.10}$$

Hence, the solution $y(t)$ (1.5*e*) of PVI is

$$y(t) = \frac{\theta_0 t f_0(t)}{(\theta_\infty - 1)\,f(t)} \tag{4.11}$$

where

$$\begin{aligned} f(t) &= -\frac{\nu_1}{2i\pi}(1 - e^{2i\pi\theta_1})\int_1^t \hat{z}^{\theta_0}(\hat{z}-1)^{\theta_1}(\hat{z}-t)^{\theta_t}\,d\hat{z} \\ f_0(t) &= \frac{\nu_1}{2i\pi}(1 - e^{2i\pi\theta_1})\int_1^t \hat{z}^{\theta_0-1}(\hat{z}-1)^{\theta_1}(\hat{z}-t)^{\theta_t}\,d\hat{z}\,. \end{aligned} \tag{4.12}$$

The functions $f(t)$ and $f_0(t)$ can be put into the form of the integral representation of the hypergeometric function and its derivative with respect to its argument [20]. Therefore, for $\theta_0 + \theta_1 + \theta_t + \theta_\infty = 0$ and for $\mathrm{Re}[\theta_0] < 1$, $\mathrm{Re}[\theta_1] > -1$, $\mathrm{Re}[\theta_t] > -1$ the solution of the PVI equation can be expressible rationally in terms of the hypergeometric function.

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